Abstract – Abstract: New approaches to the transformations of the uncontrollable and unobservable matrices of linear systems to their canonical forms are proposed. It is shown that the uncontrollable pair \((A, B)\) and unobservable pair \((A, C)\) of linear systems can be transform to their controllable \((\bar{A}, \bar{B})\) and observable \((\bar{A}, \bar{C})\) canonical forms by suitable choice of nonsingular matrix \(M\) satisfying the condition \(M [A \ B] = [\bar{A} \ \bar{B}]\) and \(A \ B]M = [A \ \bar{B}]\), respectively. It is also shown that by suitable choice of the gain matrix \(K\) of the feedbacks of the derivative of the state vector it is possible to reduce the descriptor system to the standard one.

**Key words** – controllability, observability, canonical form, descriptor, linear system.

I. **MATRIX EQUATIONS WITH NON-SQUARE MATRICES AND THEIR SOLUTIONS**

Consider the matrix equation

\[
PX = Q,
\]

where \(P \in \mathbb{R}^{m \times n}\) and \(Q \in \mathbb{R}^{r \times p}\) are given and \(X \in \mathbb{R}^{n \times p}\) is unknown matrix.

**Theorem 1.** The matrix equation (1) has a solution \(X\) if and only if

\[
\text{rank}[P \quad Q] = \text{rank}P.
\]

Proof follows immediately from the Kronecker-Cappelly Theorem [3].

**Theorem 2.** If the condition (2) is satisfied then the solution \(X\) of the equation (1) is given by

\[
X = P^+Q,
\]

where \(P^+ \in \mathbb{R}^{m \times n}\) is the right inverse of the matrix \(P\) given by

\[
P^+ = P^T \left[P P^T\right]^{-1} + (I_n - P^T \left[P P^T\right]^{-1} P)K_1, \quad K_1 \in \mathbb{R}^{n \times n}
\]

or

\[
P^+ = K_1 \left[PK_1\right]^{-1}, \quad K_1 \in \mathbb{R}^{n \times n}
\]

the matrix \(K_1\) is arbitrary and \(K_1\) is chosen so that

\[
\det\left[AK_1\right] \neq 0.
\]
There exists a nonsingular matrix $K \in \mathbb{R}^{m \times m}$, $K \neq 1$, which transforms the uncontrollable pair $(A, B)$ into their canonical forms.

**Theorem 3.** The matrix equation (7) has a solution $X$ if and only if

$$\text{rank} \begin{bmatrix} P \\ Q \end{bmatrix} = \text{rank} \begin{bmatrix} \tilde{P} \\ \tilde{Q} \end{bmatrix}.$$  

Proof is similar (dual) to the proof of Theorem 1.

**Theorem 4.** If the condition (8) is satisfied then the solution of the equation (7) is given by

$$X = \tilde{Q}P^{-1},$$

where the left inverse of the matrix $\tilde{P}$ is given by

$$\tilde{P} = \begin{bmatrix} P^T P \end{bmatrix}^{-1} P^T + K_1 (I_m - P^T P) \begin{bmatrix} P^T \end{bmatrix}^{-1} P^T, \quad K_1 \in \mathbb{R}^{m \times m}.$$  

or

$$\tilde{P} = \begin{bmatrix} K_2 \end{bmatrix}^{-1} K_2, \quad K_2 \in \mathbb{R}^{m \times m} - \text{arbitrary},$$

and the matrix $K_2$ is chosen so that $\det[K_2 \tilde{P}] \neq 0$.

Proof is similar (dual) to the proof of Theorem 2.

**II. TRANSFORMATIONS OF THE UNCONTROLLABLE PAIRS TO THEIR CANONICAL FORMS**

Consider the continuous-time linear system

$$x = Ax + Bu, \quad y = Cx,$$

where $x = x(t) \in \mathbb{R}^m$, $u = u(t) \in \mathbb{R}^n$, $y = y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{p \times m}$.

To simplify the notion we assume $m = 1$ (single input systems).

**Definition 1.** The pair $(A, B)$ is called in its canonical controllable form if

$$A_1 = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m-2} & -a_{m-1} & \ldots & -a_m \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

or

$$A_2 = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 - a_{m-1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

**Theorem 5.** There exists a nonsingular matrix $M \in \mathbb{R}^{n \times n}$ which transforms the uncontrollable pair $(A, B)$ into their canonical forms.

**Proof.** From Theorem 1 it follows that there exists a nonsingular matrix $M \in \mathbb{R}^{n \times n}$ satisfying the equation

$$M A = \tilde{A}, \quad M B = \tilde{B},$$

and the matrix $\tilde{M}$ is chosen so that $\det[\tilde{M} A] \neq 0$.

**Procedure 1.**
1. Check the condition (14). The problem has a solution if and only if the condition (14) is satisfied.
2. Using the equality $M B = \tilde{B}$ find the corresponding column of the matrix $M$.
3. Using the equality $M A = \tilde{A}$ find the remaining columns of the matrix $M$.

The theorem will be illustrated by the following simple example.

**Example 1.** Find the matrix $M \in \mathbb{R}^{2 \times 2}$ satisfying (15) which transforms the pair

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

to their canonical form

$$\tilde{A} = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$  

Using Procedure 1 we obtain the following.

**Step 1.** The condition (14) is satisfied for the matrix $\tilde{A}$ with $a_0 = 0$.

**Step 2.** From the equality $\tilde{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we obtain $m_{12} = 1, m_{22} = 0$.

**Step 3.** Taking into account (17) and...
\[ \tilde{A} = MA = \begin{bmatrix} m_{11} & 1 & 0 \\ m_{12} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \] (18)

we obtain: \( m_{11} = m_{12} = 1 \).

Therefore, the desired nonsingular matrix \( M \) has the form
\[ M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \] (19)

Remark 1. The approach based on the equation
\[ M[A B] = [\tilde{A} \tilde{B}] \] (20)
can be also used to transform the controllable pair \([A, B] \) to the desired standard controllable form \([\tilde{A}, \tilde{B}] \).

The procedure will be shown on the following simple example.

**Example 2.** For the controllable pair
\[ A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \] (21)

find the matrix \( M \in \mathbb{R}^{2 \times 2} \) satisfying the equality (20) such that the pair \([\tilde{A}, \tilde{B}] \) is the desired canonical form
\[ \tilde{A} = MA = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \tilde{B} = MB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \] (22)

From the equality
\[ \tilde{B} = MB = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \] (23)

we have \( m_{11} = 1, \ m_{12} = 0 \).

Using (22) we obtain
\[ \tilde{A} = MA = \begin{bmatrix} m_{11} & 1 \\ m_{21} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \] (24)

and \( m_{11} = -1, \ m_{12} = 1 \).

Therefore, the desired matrix \( M \) has the form
\[ M = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}. \] (25)

**III. Transformations of the Unobservable Pairs to Their Canonical Forms**

To simplify the notation we assume \( p = 1 \) (single output systems).

**Definition 2.** The pair \([A, C] \) is called in its canonical observable form if
\[ \tilde{A} = \begin{bmatrix} 0 & 0 & \ldots & -a_1 & 0 \\ 1 & 0 & \ldots & -a_2 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -a_n \end{bmatrix}, \quad \tilde{C} = [0 \ldots 0 1] \] (26a)
or
\[ \tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \ldots & -a_n \end{bmatrix}, \quad \tilde{C} = [1 \ldots 0 0] \] (26b)

**Theorem 6.** There exists a nonsingular matrix \( M \in \mathbb{R}^{n \times n} \) which transforms the unobservable pair \([A, C] \) satisfying the condition
\[ \text{rank} \begin{bmatrix} A \\ C \end{bmatrix} = n \] (27)
to the canonical forms (26) if and only if
\[ \text{rank} \begin{bmatrix} A \\ C \tilde{C} \end{bmatrix} = \text{rank} \begin{bmatrix} \tilde{A} \\ C \end{bmatrix} \] (28)

for \( k = 1, 2 \).

Proof is similar (dual) to the proof of Theorem 5.

If the condition (28) is satisfied then for the given matrices \( A, C \) and \( \tilde{A}, \tilde{C} \), the matrix \( \tilde{M} \) can be computed by the use of the following procedure.

**Procedure 2.**

**Step 1.** Check the condition (28). The problem has a solution if and only if the condition (28) is satisfied.

**Step 2.** Using the equality \( CM = \tilde{C} \) find the corresponding column of the matrix \( \tilde{M} \).

**Step 3.** Using the equality \( AM = \tilde{A} \) find the remaining columns of the matrix \( \tilde{M} \).

The procedure will be illustrated by the following simple example.

**Example 3.** Find the matrix \( \tilde{M} \in \mathbb{R}^{2 \times 2} \) satisfying (28) which transforms the pair
\[ A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad C = [1 0] \] (29)
to their canonical form
\[ \tilde{A} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}, \quad \tilde{C} = [0 1]. \] (30)

Using Procedure 2 we obtain the following.

**Step 1.** The condition (28) is satisfied since
\[ \text{rank} \begin{bmatrix} A \\ C \tilde{C} \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & -3 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \text{rank} \begin{bmatrix} A \\ C \end{bmatrix}. \] (31)

**Step 2.** From the equality
\[ CM = [1 0] \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = [0 1] \] (32)

we obtain: \( m_{11} = 0, \ m_{12} = 1 \).

**Step 3.** Taking into account (29) and

\[ C = [0 1] \] (33)
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\[ A\hat{M} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \]  

(33)

we obtain: \( m_{21} = 1 \), \( m_{22} = -2 \).

Therefore, the desired nonsingular matrix \( \hat{M} \) has the form

\[ \hat{M} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}. \]  

(34)

IV. TRANSFORMATIONS OF THE CONTROLLABLE PAIRS TO THEIR CANONICAL FORMS

Consider the following two pairs \( (\hat{A}, \hat{B}) \) and \( (\hat{A}, \hat{B}) \) in canonical forms (26). We are looking for nonsingular matrix \( M \in \mathbb{R}^{mn \times mn} \) such that

\[ M(\hat{A}, \hat{B}) = [A \quad B]. \]  

(35)

**Theorem 7.** The pair \( (\hat{A}, \hat{B}) \) can be transformed into \( (A, B) \) if and only if

\[ \text{rank} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{A} & \hat{B} \end{bmatrix} = \text{rank} \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix}. \]  

(36)

**Proof.** By Theorem 1 the equation (35) has a solution \( M \) if and only if the condition (36) is satisfied. \( \square \)

Now we apply Theorem 7 to the pair \( (\hat{A}, \hat{B}) \) in their canonical form (12) and we obtain the following theorem.

**Theorem 8.** The pair (12a) cannot be transformed into pair (12b) by the nonsingular matrix \( M \in \mathbb{R}^{n,m} \) satisfying (35).

**Proof.** Applying the condition (36) to the pairs (12) we obtain

\[ \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & -a_1 \\ 1 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 - a_{n+1} \\ \end{bmatrix} = \begin{bmatrix} -a_n & -a_{n+1} & -a_{n+2} & \ldots & -a_{n+1} \\ 0 & 0 & 0 & \ldots & -a_1 \\ 1 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 1 - a_{n+1} \\ \end{bmatrix} \]  

(37)

\[ \text{rank} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{A} & \hat{B} \end{bmatrix} = \text{rank} \begin{bmatrix} A \\ B \end{bmatrix}. \]

and

\[ \text{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \text{rank} \begin{bmatrix} E \\ F \end{bmatrix}. \]  

(44)

**V. ELIMINATION OF THE SINGULARITY IN DESCRIPTOR LINEAR SYSTEMS**

Consider the continuous-time linear system

\[ E\dot{x} = Ax + Bu, \]  

(45)

where \( x = x(t) \in \mathbb{R}^n \), \( u = u(t) \in \mathbb{R}^m \) are the state, input and output vectors and \( E, A \in \mathbb{R}^{mn \times mn} \).

It is assumed that \( \det E = 0 \) and \( \det(Es - A) \neq 0, s \in \mathbb{C} \) (the field of complex numbers). (46)

We are looking for the matrix \( M \in \mathbb{R}^{mn \times mn} \) satisfying the equality

\[ M[E \quad A \quad B] = [I_m \quad \hat{A} \quad \hat{B}] \]  

(47)

which eliminate the singularity of the system (45). Note that by Theorem 1 there exists the nonsingular matrix \( M \) satisfying (47) if and only if the condition

\[ \text{rank} \begin{bmatrix} E \\ A \end{bmatrix} = \text{rank} \begin{bmatrix} E \quad A \quad B \end{bmatrix}. \]  

(48)
is satisfied. From (47) we have
\[ ME = I_k. \]  
(49)

Note that for nonsingular matrix \( M \) the equation (49) has no singular solution \( E \).

Therefore we have the following conclusion. By suitable choice of the matrix \( M \) it is not possible to transform the descriptor system (45) to the standard one of the form
\[ \dot{x} = \bar{A}x + \bar{B}u, \]  
(50)

where \( x = x(t) \in \mathbb{R}^n \), \( u = u(t) \in \mathbb{R}^m \) are the state, input and output vectors and \( \bar{A} \in \mathbb{R}^{n \times n} \), \( \bar{B} \in \mathbb{R}^{m \times n} \).

VI. REDUCTION OF THE DESCRIPTOR LINEAR SYSTEMS TO
STANDARD ONES BY FEEDBACKS

Consider the descriptor system (45) with feedbacks of the derivative of the state vector shown in Fig.1.

Fig.1. Descriptor system with feedback

Substituting the equality
\[ u = v - K \dot{x} \text{ (the new input)} \]  
(51)

into the equation
\[ \dot{E}x = Ax + Bu \]  
(52)

we obtain
\[ \dot{E} \dot{x} = Ax + B(v - K \dot{x}) \]  
(53)

and
\[ (E + BK) \dot{x} = Ax + Bv. \]  
(54)

The feedback matrix \( K \in \mathbb{R}^{m \times n} \) is chosen so that the matrix
\[ F = E + BK = [E \quad B] \begin{bmatrix} I_k \vline \end{bmatrix} \]  
(55)

is nonsingular.

Note that there exists the matrix \( K \) such that the matrix \( F \) is nonsingular if and only if
\[ \text{rank} \begin{bmatrix} E \quad B \end{bmatrix} = n. \]  
(56)

Note that the equation (55) by Theorem 1 has the solution \( \begin{bmatrix} I_k \vline \end{bmatrix} \) if and only if
\[ \text{rank} \begin{bmatrix} E \quad B \quad F \end{bmatrix} = \text{rank} \begin{bmatrix} E \quad B \end{bmatrix} \]  
(57)

and this condition is satisfied if and only if (56) holds.

Therefore, the following theorem has been proved.

**Theorem 10.** There exists the matrix \( K \) such that the matrix \( F \) is nonsingular if and only if the condition (56) is satisfied.

For nonsingular matrix \( F \) from (54) we have
\[ \dot{x} = \bar{A}x + \bar{B}v, \]  
(58)

where
\[ \bar{A} = F^{-1}A, \quad \bar{B} = F^{-1}B. \]  
(59)

**Example 4.** Consider the system (52) with the matrices
\[ E = \begin{bmatrix} 0 & 1 \\ \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]  
(60)

which satisfies the condition (56) since
\[ \text{rank} \begin{bmatrix} E \quad B \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 2. \]  
(61)

By Theorem 10 there exists the feedback matrix \( K = [k_1 \quad k_2] \) such that the matrix
\[ F = E + BK = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix} \]  
(62)

is nonsingular. In this case the matrix (62) is nonsingular if \( k_2 = 0 \) and \( k_1 \) is nonzero. For \( k_1 = 1, k_2 = 0 \) we have
\[ K = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]  
(63)

and
\[ \bar{A} = F^{-1}A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \]  
(64)

\[ \bar{B} = F^{-1}B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \quad 1 \\ 1 \quad 0 \end{bmatrix}. \]  
(65)

VII. CONCLUSIONS

Two approaches to the transformations of the uncontrollable and unobservable linear systems to their canonical forms has been proposed (Theorems 5 and 6) and procedures for calculation of transformation matrices have been given (Procedures 1 and 2). The procedures have been illustrated by simple numerical examples. It has been shown that the pair (12a) cannot be transformed to the pair (12b) by the nonsingular matrix \( M \) satisfying (55) (Theorem 7). Necessary and sufficient conditions have been established for the reduction of the descriptor linear systems to their standard forms (Theorem 8). The considerations can be extended to the discrete-time linear systems and to the fractional orders linear systems. An open problem is an extension of these approaches to the different orders linear systems.

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**Wybrane zagadnienia analizy układów liniowych**

Zaproponowano nowe podejścia do transformacji niestabilnych i nieobserwowych macierzy układów liniowych do ich postaci kanonicznej. Wykazano, że niestabilna para \((A_0, B_0)\) i nieobserwowa para \((A_0, C_0)\) układów liniowych może być przekształcona do ich postaci kanonicznych stereowalnych i obserwowych przez odpowiedni dobór nieskończonej macierzy \(M\) spełniającej warunki

\[
M[A \ B] = [\hat{A} \hat{B}]\quad [A \ B]M = [\hat{A} \hat{B}].
\]

Pokażano, że przez odpowiedni dobór macierzy \(M\) przekształcenie zwrotnego od podobnej wektora stanu jest możliwa redukcja układu desluktorowego do układu standardowego.

**Streszczenie:** sterowalność, obserwowalność, postać kanoniczna, układ desluktorowy, układ liniowy

**Bibliografia**