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Two component modified Lilliefors test for normality

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Abstract

Research background: Commonly known and used parametric tests e.g. Student, Behrens–Fisher, Snedecor, Bartlett, Cochran, Hartley tests are applicable when there is an evidence that samples come from the Normal general population. What makes things worse is that testers are not fully aware in what degree of abnormality distorts results of parametric tests listed above and suchlike. So, it is no exaggeration to say that testing for normality (goodness-of-fit testing, GoFT) is a gate to proper parametric statistical reasoning. It seems that the gate opens too easily. In other words, most popular goodness-of-fit tests are weaker than statisticians want them to be.

Purpose of the article: The main purpose of this paper is to put forward the GoFT that is, in particular circumstances, more powerful than GoFTs used until now. The other goals are to define a similarity measure between an alternative distribution and the normal one and to calculate the power of normality tests for a big set of alternatives. And, of course, to interest statisticians in using the GoFTs in their practice.

Method: There are two ways to make GoFT more powerful: extensive and intensive one. The extensive method consists in drawing large samples. The intensive method consists in extracting more information from small samples. In order to make the test method intensive, the test statistics, as distinct from all existing GoFTs, has two components. The first component (denoted by Δ) is a classic Kolmogorov / Lilliefors test statistics i.e. the greatest absolute difference between theoretical and empirical cumulative distribution functions. The second component is the order statistics (r) at which the $\Delta_{\max}((r))$ locate itself. Of course $\Delta_{\max}((r))$ is the conditional random variable with (r) being the condition. Large scale Monte Carlo simulations provided data sufficient to in-depth study of properties of distributions of $\Delta_{\max}((r))$ random variable.

Findings & value-added: Simulation study shows that the Two Component Modified Lilliefors test for normality is the most powerful for some type of alternatives, especially for the symmetrical, unimodal and bimodal distributions with positive excess kurtosis, for symmetrical and unimodal distributions with negative excess kurtosis and small sample sizes. Due to the values of

skewness and excess kurtosis, and the defined similarity measure between the ND and an alternative, alternative distributions are close to the normal distribution. Numerous examples of real data show the usefulness of the proposed GoFT.

Introduction

One cannot disagree with the statement that the most common package of statistical inference procedures are applied to the normal distribution parameters. However, before analysts use one of these procedures, they need to check whether the empirical data come from the general population in which the normal distribution “is in force”. In other words, analysts are required to perform a goodness-of-fit test (GoFT). As shown in the next section, many authors have proposed GoFTs for normality in recent years. Their goal was to define the most powerful GoFT. The goal of the author of this work is to join a large family of authors of this type of tests.

Any GoFT has its unique measure of discrepancy between theoretical and sample distributions, commonly called the test statistics. In the pre-Monte Carlo era inventors of GoFTs had their hands tied. Formulas of test statistics had to be simple. This simplicity was an indispensable condition for test statistics distributions, which were to be derived in an analytical way. It was the only way they had. In the Monte Carlo era, inventors of GoFTs have their hands freed thanks to Lilliefors.

The first goal of the paper is to put into practice the Two Component Modified Lilliefors test for normality. The first component is the greatest absolute value of a difference between CDF and EDF. The second component is the order statistics at which the greatest difference is located. The second goal is to define a similarity measure between an alternative distribution and the normal one. The third goal is to calculate the power of normality tests for a big set of alternatives. The implementation of the new proposal by means of five real data sets is presented.

The other sections of the paper are as follows. Section 2 provides a literature review on GoFTs for normality. Section 3 describes modified Lilliefors goodness-of-fit test for normality with two-component test statistics and presents the family of non-normal (alternative) distributions. Section 4 shows how particular tests vary in their power and evaluates the performance of the proposed test based on real-life examples. Section 5 is dedicated to the discussion. The paper ends with conclusions. The R codes are provided in Appendix and the notation used throughout the paper is presented in Table 0.

Literature review

Many GoFTs for normality are discussed in the statistical literature. In recent years, several new papers have been devoted to the GoFTs, e.g.: Yazici and Yolacan (2007), Romao *et al.* (2010), Razali and Wah (2011), Alizadeh Noughabi and Arghami (2011), Yap and Sim (2011), Ahmad and Khan (2015), Torabi *et al.* (2016), Marange and Qin (2019) and Sulewski (2019a, 2019b).

Some very popular normality testing procedures available in the statistical software are: the KS test (Smirnov, 1948), the LF test (Lilliefors, 1967), the CVM test (Cramér, 1928), the AD test (Anderson & Darling, 1952) and the SW test (Shapiro & Wilk, 1965). The first four belong to the EDF tests.

Let $f(x)$ be a continuous PDF on R , $F(x)$ be the corresponding CDF and $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ is a sample of size n . The EDF is defined as

$$F_e(x) = \frac{1}{n} \sum_{i=1}^n \theta(x - x_i), \quad (1)$$

where $\theta(x) = 1$ ($x \geq 0$) and $\theta(x) = 0$ ($x < 0$). Then the KS statistic is defined by

$$KS = \max_x |F_e(x) - F(x)|.$$

The δ -corrected KS test (Harter *et al.*, 1984), also studied by Khamis (1990, 1992, 1993), redefines the EDF value at the data points and compares the redefined EDF to the CDF at the data points. Let's suppose $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ is the ordered sample of size n and the EDF at the i -th data point has the form

$$F_\delta(x_{(i)}) = \frac{i - \delta}{n - 2\delta + 1}, \quad 0 \leq \delta \leq 1. \quad (2)$$

The δ -corrected KS statistic is given by

$$KS_\delta = \max_{i=1, \dots, n} |F_\delta(x_{(i)}) - F(x_{(i)})|.$$

Harter *et al.* (1984) presents the following EDF definitions:

- $F_0(x_{(i)}) = i/(n + 1)$ for $\delta = 0$
- $F_{0.5}(x_{(i)}) = (i - 0.5)/n$ for $\delta = 0.5$
- $F_1(x_{(i)}) = (i - 1)/(n - 1)$ for $\delta = 1$

- For $\delta = 0.375$ (Bloom, 1958) we have

$$F_{0.375}(x_{(i)}) = (i - 0.375)/(n + 0.25). \quad (3)$$

- For $\delta = 0.3175$ (Filliben, 1975) we have

$$F_{0.3175}(x_{(i)}) = (i - 0.3175)/(n + 0.365). \quad (4)$$

$F_{0.375}(x_{(i)})$ (3) is approximating the mean of i -th order statistic of the standard normal distribution, while (4) is approximating the median. The δ -corrected KS test using (3) and (4) - according to D'Agostino and Stephens (1986) — is characterized by weak power.

Blom (1958) defined the $\bar{\alpha}, \bar{\beta}$ transformation

$$F_{\bar{\alpha}, \bar{\beta}}(x_{(i)}) = \frac{i - \bar{\alpha}}{n - \bar{\alpha} - \bar{\beta} + 1}, \quad \bar{\alpha}, \bar{\beta} \leq 1 \quad (5)$$

to reduce the mean squared error (MSE) of some statistics. This transformation can be used to create the GoFT.

Feltz (2002), based on the paper of Blom (1958), defined the $(\bar{\alpha}, \bar{\beta})$ -corrected KS test. The $(\bar{\alpha}, \bar{\beta})$ -corrected KS statistic is as follows

$$KS_{\bar{\alpha}, \bar{\beta}} = \max_{i=1, \dots, n} |F_{\bar{\alpha}, \bar{\beta}}(x_i) - F(x_i)|,$$

where $F_{\bar{\alpha}, \bar{\beta}}(x)$ is the EDF given by (5). If $\bar{\alpha} = \bar{\beta} = \delta$, then the δ -corrected KS test is a special case of the $(\bar{\alpha}, \bar{\beta})$ -corrected KS test. We can analyze, using (5), a bigger family of modified KS statistics.

The family of the δ -corrected KS tests is smaller than the family of the $(\bar{\alpha}, \bar{\beta})$ -corrected KS tests and, therefore, we can define more powerful tests than the δ -corrected KS test. Feltz (2002) studied δ -corrected KS tests for $\delta = 0, 1$ and H_0 : data are from $N(0, 1)$, H_1 : data are from $\omega N(-\mu, \sigma) + (1 - \omega)N(\mu, \sigma)$. He, varying μ, σ, ω values, obtained three groups of skewness (g_1) and excess kurtosis (g_2): $g_1 > 0$ and $g_2 > 0$, $g_1 = 0$ and $g_2 < 0$, $g_1 > 0$ and $g_2 < 0$.

Sulewski (2019b) studied the $(\bar{\alpha}, \bar{\beta})$ -corrected KS test for normality (denoted as $LF(\bar{\alpha}, \bar{\beta})$) using $\bar{\alpha}$ and $\bar{\beta}$ values presented in Table 1. The test statistic is defined as

$$LF_{\bar{\alpha}, \bar{\beta}} = \max_{i=1, \dots, n} |F_{\bar{\alpha}, \bar{\beta}}(x_{(i)}) - \Phi(x_{(i)}, \bar{x}, s)|, \quad (6)$$

where $F_{\bar{\alpha}, \bar{\beta}}(x)$ is given by (5), $\bar{x}$ is the sample mean and s is the sample standard deviation with denominator $(n - 1)$. The $LF(0,1)$ test is the most powerful for the alternatives with $g_1 > 0$, the $LF(1,0)$ test - for the alternatives with $g_1 < 0$, the $LF(1,1)$ test — for the alternatives with $g_1 = 0$ and $g_2 > 0$, the $LF(0,0)$ test — for the alternatives with $g_1 = 0$ and $g_2 < 0$.

Research methodology

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be a sample of size n . The Lilliefors test statistics for normality is given by (Lilliefors, 1967)

$$LF = \max_{i=1, \dots, n} |F_e(x_{(i)}) - \Phi(x_{(i)}, \bar{x}, s)|,$$

where $F_e(x)$ is the EDF, $\bar{x}$ is the sample mean and s is the sample standard deviation with denominator $(n - 1)$. The LF test for normality is meant for testing H_0 : data come from the ND and H_1 : not H_0 . Hypothesis H_0 is rejected when LF value exceeds the critical value.

The Modified Lilliefors test for normality is defined by the test statistic (6). For the needs of the paper, let's call it One Component Modified Lilliefors Test for Normality.

Two Component Modified Lilliefors test for normality

This section deals with the first aim of the paper. The Two Component Modified Lilliefors test with EDF (5) test denoted by $TCLF(\bar{\alpha}, \bar{\beta})$ has two components. The first component (denoted by Δ) is, like in the original LF test, an absolute value of the greatest discrepancy between sample and population distributions. The second component (denoted by r) is a position in an ordered sample at which this discrepancy is located. Both components, Δ and r , are random variables. The conditional distributions of the random variable Δ_r given $r = r^*, r^* = 1, 2, \dots, n$, where n is a sample size, were determined with the Monte Carlo method. Then, this distribution served as the basis for determining conditional critical values Δ_r^α . The decision rule remains unchanged. The H_0 is accepted when $\Delta_{r^*} < \Delta_{r^*}^\alpha$. The $TCLF(\bar{\alpha}, \bar{\beta})$ test statistics is defined for $\bar{\alpha}, \bar{\beta} \leq 1$ as

$$\Delta_r = \max_{i=1, \dots, n} \left| F_{\bar{\alpha}, \bar{\beta}}(x_{(i)}) - \Phi\left(\frac{x_{(i)} - \bar{x}}{s}, 0, 1\right) \right|, \quad (7)$$

where $F_{\bar{\alpha}, \bar{\beta}}(x)$ is given by (5), $\bar{x}$ is the sample mean, s is the sample standard deviation with denominator $(n - 1)$ and r is the order statistics at which an absolute value of the greatest difference between CDF of the $N(0,1)$ and EDF is located. The $TCLF(\bar{\alpha}, \bar{\beta})$ test for normality is meant for testing H_0 : data come from the ND with PDF $\phi(x; 0, 1)$ and H_1 : not H_0 . The Monte Carlo simulations use the $\bar{\alpha}, \bar{\beta}$ values presented in Table 1 (see Sulewski, 2019b).

Alternative distributions

The second goal of the paper is to define a similarity measure between the ND and alternative distribution. A set of alternative distributions considered in this paper comprises 60 distributions further called alternatives. Alternatives used in Monte Carlo simulations, based on the inequality $\gamma_2 \geq \gamma_1 - 2$ (Malachov, 1978), are presented in groups A1-F2 (see Table 2).

The first alternative is the CND($\mu_1, \sigma_1, \mu_2, \sigma_2, \omega$) with PDF

$$f_{CND}(x, \mu_1, \sigma_1, \mu_2, \sigma_2, \omega) = \omega \phi(x; \mu_1, \sigma_1) + (1 - \omega) \phi(x; \mu_2, \sigma_2).$$

One can obtain skewness and excess kurtosis values belonging to each of the analyzed group (except C2) by selecting appropriate parameter values of the CND.

Other alternatives are presented in Table 3. Among them, there are distributions: studied in Gan and Koehler (1990), Esteban *et al.* (2001), Krauczi (2009) and Torabi *et al.* (2016); selected by the author (*in italic*) and defined by the author (*in bold*).

The alternatives are defined for negative and no-negative real numbers. Their domain, except uniform distribution and truncated $N(0,1)$ one, is real numbers. Table 4 presents the alternatives used in the Monte Carlo simulation, divided into twelve groups and numbered from 1 to 60. The author chose the alternatives and values of their parameter so that they were close to the ND. Two measures were used to choose their parameters. The first one is the skewness and excess kurtosis. The other one is the similarity measure between the ND and an alternative. This measure is defined based on the test statistic (7) and is given by

$$M = \max_x |F(x) - \Phi(x, a, b)|,$$

where $F(x)$ is the CDF of an alternative with mean a and standard deviation b . Obviously, the smaller the value of M , the bigger the alternative similarity to the ND. The M values for all the alternatives are presented in Table 4.

Results

Power comparisons

The third goal of the paper is to calculate the PoTs for 60 alternatives. Sample sizes are typical for statistical literature, namely, $n = 20$ patterned after Harter *et al.* (1984), Torabi *et al.* (2016), Sulewski (2019b) and $n = 50, 100$ patterned after (Romao *et al.*, 2010; Yap & Sim, 2011).

Summing up, two hypotheses were formulated, namely H_0 and H_1 . Hypothesis H_0 states that data come from the general population where the ND holds good. Hypothesis H_1 , in turn, contradicts H_0 . It states that data come from the general population where non-normal distribution holds good. A set of alternatives considered in this paper comprises 60 distributions. The alternatives were listed in Table 4.

A collection of Δ_r^α critical values ($r = 1, 2, \dots, n$) for the type I error α is determined on the basis of $m_r = 10^4$ Monte Carlo experiments. One experiment consisted of generating $n = 20, 50, 100$ data points from the $N(0, 1)$ and calculating values of the test statistics (7) for seven EDFs from Table 1. The r -th ($r = 1, \dots, n$) order statistics for which there is the greatest absolute value of difference between CDF of the $N(0, 1)$ and EDF is determined for each Monte Carlo experiment. Critical values for extreme r values (in bold, see Table 5) are determined by using $m_r = 10^3$ experiments because the probability of occurrence of the extreme order statistics is very small. Unknown quantile values are replaced by appropriate order statistics $cv_{0.05} = Q_{([0.95m_r])}$. Table 5 presents critical values $\Delta_r^{0.05}$ for the $TCLF(\bar{\alpha}, \bar{\beta})$ test for $n = 20$.

Table 6 presents the actual size of type 1 error rate at $\alpha = 0.05$ for the proposed GoFT. These results were obtained from 10^5 experiments.

Tables 7–13 present the power of the $TCLF(\bar{\alpha}, \bar{\beta})$ and $LF(\bar{\alpha}, \bar{\beta})$ tests calculated for a big set of the alternatives from Table 4 and $\bar{\alpha}, \bar{\beta}$ values from Table 1. PoTs are obtained from 10^5 experiments. The highest PoTs for the discussed GoFTs are in bold. As it was mentioned in Sulewski (2019b), $\bar{\alpha}$ and $\bar{\beta}$ values depend on the nature of an alternative that seems to pass for the ND. The $\bar{\alpha} = 0$ and $\bar{\beta} = 1$ (i.e. the EDF is i/n) for the alter-

natives with positive skewness. The $\bar{\alpha} = 1$ and $\bar{\beta} = 0$ (i.e. the EDF is $(i - 1)/n$) for the alternatives with negative skewness. The bimodality of the alternatives does not affect the choice of the $\bar{\alpha}$ and $\bar{\beta}$ values.

In order to contrast $TCLF(\bar{\alpha}, \bar{\beta})$ test with $LF(\bar{\alpha}, \bar{\beta})$ test in view of their power Table 14 was constructed. The symbol $N(\bar{\alpha}, \bar{\beta})$ (in bold) means that the power of the $TCLF(\bar{\alpha}, \bar{\beta})$ test is higher by at least 0.002 than the power of the $LF(\bar{\alpha}, \bar{\beta})$ test. The symbol $O(\bar{\alpha}, \bar{\beta})$ means the opposite situation. The symbol $ON(\bar{\alpha}, \bar{\beta})$ means that the difference between the PoTs is less than 0.002. The PoTs differ by less than 0.002 for 6 alternatives for $n = 20$, 7 alternatives for $n = 50$, 15 alternatives for $n = 100$. Thus, the higher the sample size, the more difficult it is to select the powerful test, which obviously was expected. The new GoFT is the most powerful for alternatives numbered 14, 36 ($n = 100$), 34 ($n = 20, 100$), 5, 17, 21–29, ($n = 50, 100$), 32, 35, 57 ($n = 20, 50, 100$). The Monte Carlo results show (see too D’Agostino and Stephens (1986)), that the $LF(3/8, 3/8)$ test is characterized by a weak power. The similar applies to the $TCLF(3/8, 3/8)$ test.

Numerical examples

This Section is devoted to applying the $TCLF(\bar{\alpha}, \bar{\beta})$ test with $\bar{\alpha}$ and $\bar{\beta}$ values from Table 1 (Sulewski, 2019b) using five real data sets. A new proposal is compared with very popular GoFTs for normality such as AD, CVM, SW, Shapiro-Francia (Shapiro & Francia, 1972), χ^2 and GoFTs developed in the last five years such as Hn (Torabi *et al.*, 2016), MAD (Sulewski, 2019a), $LF(\bar{\alpha}, \bar{\beta})$ test with $\bar{\alpha}$ and $\bar{\beta}$ values from Table 1 (Sulewski, 2019b).

Example I studies the height of 99 five-year-old British boys (cm) from <http://www.mas.ncl.ac.uk/~njnsm/medfac/docs/intro.pdf>. Example II presents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows (gauge lengths of 20 mm) (Kundu & Raqab, 2009; Nofal *et al.*, 2017). Example III presents the average weights for American women aged 30–39 (R package “women”). Example IV presents the number of people who have been arrested for assault per 100,000 residents in each of the 50 US states (R package “USArrests”). Example V presents the percent of the population living in urban areas in each of the 50 US states (R package “USArrests”). The data sets in Appendix are presented. The sample skewness and the sample excess kurtosis are: $g_1 = -0.109$, $g_2 = -0.072$ (example I); $g_1 = -0.157$, $g_2 = 0.033$ (example II), $g_1 = 0.281$, $g_2 = -1.041$ (example III), $g_1 = 0.234$, $g_2 = -1.054$ (example IV), $g_1 = -0.226$, $g_2 = -0.738$ (example V).

Figure 1 presents the histograms of these data sets after standardization. We calculate p-values for the discussed GoFTs based on 10^5 experiments when fitting the $N(0,1)$ to the data (see Tables 15 and 16). The smallest p-value is in bold. The non-normality in Example I is the most pronounced by the $TCLF(1,0)$, $TCLF(1,1)$ and $LF(1,0)$ tests. The non-normality in Example II is the most pronounced by the $TCLF(0,1)$, $TCLF(1,1)$ and $LF(0,1)$ tests. The non-normality in Example III is the most pronounced by the $TCLF(1,1)$, MAD and SW tests. The non-normality in Example IV is the most pronounced by the $TCLF(0,1)$, $LF(0,1)$ and $LF(0,0)$ tests. The non-normality in Example V is the most pronounced by the $TCLF(1,0)$, $TCLF(0,0)$ and $LF(1,0)$ tests.

Appendix presents R codes to calculate the $TCLF(\bar{\alpha}, \bar{\beta})$ test statistic with r position, the corresponding p-value and the critical value for $B = 10^4$ experiments.

Discussion

The first goal of the paper is to put into practice the Two Component GoFT for normality. Critical values of the proposed test are determined for position $r = 1, 2, \dots, n$ of discrepancy between theoretical and empirical cumulative distribution functions. Due to values of the similarity measure between two distributions (the second goal), the alternatives divided in twelve groups according to skewness and excess kurtosis values are close to the ND.

The $TCLF(1,1)$ test is the most powerful for all the analyzed symmetrical, unimodal and bimodal distributions with positive excess kurtosis (groups C1 and C2) and for the alternative numbered 5 for the sample size $n = 50, 100$. The $TCLF(0,1)$ is the most powerful for all the analyzed (except alternative 30), symmetrical and unimodal distributions with negative excess kurtosis (group D1) for the sample size $n = 20$. The $TCLF(0,0)$ is the most powerful for the selected, symmetrical and unimodal distributions with negative excess kurtosis (group D1) for sample size $n = 50, 100$. The $TCLF(1,0)$ test is the most powerful for the alternatives 17 ($n = 50, 100$) and 57 ($n = 20, 50, 100$). The $TCLF(0,1)$ test is recommended for the selected alternatives from group A2, especially for higher sample sizes. The $TCLF(1,0)$ test is recommended for the selected alternatives from group B2, especially for higher sample sizes.

Conclusions

To sum up, the proposed test, as mentioned in the previous section, is the most powerful for some alternatives. The simulation study, as could be expected (Janssen, 2000), shows that none of the examined tests can be considered as the best for all the alternatives. The good performance of the $TCLF(\bar{\alpha}, \bar{\beta})$ test against other most popular GoFTs through the analysis of real data sets is illustrated. The Monte Carlo simulations are carried out for 60 alternatives. Such a large number of distributions is used to ensure reliable results.

Is the $TCLF(\bar{\alpha}, \bar{\beta})$ for exponentiality also noteworthy? The author of this paper intends to answer this question in the near future.

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Annex

Table 0. The notation used throughout the paper

Notation	Description	Notation	Description
CVM	Cramer-von Mises	KS	Kolmogorov-Smirnov
PDF	Probability density function	SW	Shapiro-Wilk
CDF	Cumulative distribution function	LF	Lilliefors
EDF	Empirical distribution function	GoFT	Goodness-of-fit test
ND	Normal distribution	$\phi(x, a, b)$	PDF of ND
CND	Compound normal distribution	$\Phi(x, a, b)$	CDF of ND
AD	Anderson-Darling	PoT	Power of test

Table 1. EDF definitions in the $TCLF(\bar{\alpha}, \bar{\beta})$ test

$(0,1)$	$(1,0)$	$(0,0)$	$(0.5,0.5)$	$(0.3,0.3)$	$(1,1)$	$(0.375,0.375)$
$\frac{i}{n}$	$\frac{i-1}{n}$	$\frac{i}{n+1}$	$\frac{i-0.5}{n}$	$\frac{i-0.3}{n+0.4}$	$\frac{i-1}{n-1}$	$\frac{i-3/8}{n+1/4}$

Source: Sulewski (2019b).

Table 2. Alternative distributions according to groups A1-F2

Group	g_1	g_2	Group	g_1	g_2
A1*	+	+	D1*	0	-
A2**			D2**		
B1*	-	+	E1*	+	-
B2**			E2**		
C1*	0	+	F1*	-	-
C2**			F2*		

* unimodal distribution, ** bimodal distribution

Source: Sulewski (2019b).

Table 3. Alternatives used in the Monte Carlo simulation

Name of alternative distribution	PDF of alternative distribution
Gumbel - $GU(\mu, \sigma)$	$f_{GU}(x; \mu, \sigma) = \frac{1}{\sigma} \exp\left[\frac{\mu-x}{\sigma} - \exp\left(\frac{\mu-x}{\sigma}\right)\right]$
Compound Gumbel - $CGU(A_1)$	$f_{CGU}(x; A_1) = \omega f_{GU}(x; \mu_1, \sigma_1) + (1-\omega) f_{GU}(x; \mu_2, \sigma_2)$
Normal Gumbel mixture - $NGUM(A_1)$	$f_{NGUM}(x; A_1) = \omega \phi(x; \mu_1, \sigma_1) + (1-\omega) f_{GU}(x; \mu_2, \sigma_2)$
Normal Weibull mixture - $NWM(\mu, \sigma, c, \omega)$	$f_{NWM}(x; \mu, \sigma, c, \omega) = \omega \phi(x; \mu, \sigma) + (1-\omega) f_W(x; c)$

Table 3. Continued

Name of alternative distribution	PDF of alternative distribution
Skew-normal mixture – $SNM(A_3)$	$f_{SNM}(x; A_3) = \omega f_{sk}(x; \mu_1, \sigma_1, c_1) + (1 - \omega) f_{sk}(x; \mu_2, \sigma_2, c_2)$ $f_{sk}(x; \mu, \sigma, c) = \frac{2}{\sigma} \phi\left(\frac{x - \mu}{\sigma}, 0, 1\right) \Phi\left(c \frac{x - \mu}{\sigma}, 0, 1\right)$
Johnson's $S_U - SU(\mu, \sigma)$	$f_{SU}(x; \mu, \sigma) = \frac{\sqrt{x^2 + 1}}{\sigma \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{\ln(x + \sqrt{x^2 + 1}) - \mu}{\sigma}\right)^2\right]$ $x \geq 0$
Laplace – $LAP(\mu, \sigma)$	$f_L(x; \mu, \sigma) = \frac{1}{2\sigma} \exp\left(-\frac{ x - \mu }{\sigma}\right)$
Compound Laplace – $CL(A_1)$	$f_{CL}(x; A_1) = \omega f_L(x; \mu_1, \sigma_1) + (1 - \omega) f_L(x; \mu_2, \sigma_2)$
Normal Laplace mixture – $NLM(A_1)$	$f_{NLM}(x; A_1) = \omega \phi(x; \mu_1, \sigma_1) + (1 - \omega) f_L(x; \mu_2, \sigma_2)$
Student's $t - t(n)$	$f_t(x; n) = (n\pi)^{-0.5} \Gamma^{-1}\left(\frac{n}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \left(1 + \frac{x^2}{n}\right)^{(-n-1)/2}$
Uniform – $U(a, b)$	$f_U(x; a, b) = \frac{1}{b-a} \quad a \leq x \leq b$
Truncated $N(0,1) - TSN(a, b)$	$f_{TSN}(x; a, b) = \frac{\phi(x, 0, 1)}{\Phi(b, 0, 1) - \Phi(a, 0, 1)}$
ND with plasticizing component – $NDPC(A_2)^B$	$f_{NDPC}(x; A_2) = \omega \phi(x; \mu_1, \sigma_1) + (1 - \omega) f_{PC}(x; \mu_2, \sigma_2, c)$ $f_{PC}(x; \mu, \sigma, c) = \frac{c}{\sigma \sqrt{2\pi}} \left \frac{x - \mu}{\sigma}\right ^{c-1} \exp\left[-\frac{1}{2} \left \frac{x - \mu}{\sigma}\right ^{2c}\right] c \geq 1$
Compound plasticizing component – $CPC(A_3)^B$	$f_{CPC}(x; A_3) = \omega f_{PC}(x; \mu_1, \sigma_1, c_1) + (1 - \omega) f_{PC}(x; \mu_2, \sigma_2, c)$

^B Main component of these distribution is the plasticizing component (PC) with PDF $f_{PC}(x; \mu, \sigma, c)$ which is briefly described in (Sulewski 2019b). The $NDPC(\mu_1, \sigma_1, \mu_2, \sigma_2, c, \omega)$ is described in (Sulewski, 2020).

Notation:

$$A_1 = \mu_1, \sigma_1, \mu_2, \sigma_2, \omega; A_2 = \mu_1, \sigma_1, \mu_2, \sigma_2, c_2, \omega; A_3 = \mu_1, \sigma_1, c_1, \mu_2, \sigma_2, c_2, \omega.$$

Table 4. Alternatives for groups A1–F2

No.	Distribution	Skewness	Ex. kurtosis	Group	M
1	$CGU(0,1,2,1,0.8)$	0.918	1.263	A1	0.065
2	$NDPC(1,1,0,0.5,1,0.5)$	0.687	0.388		0.075
3	$CND(0,1,2,3,0.7)$	1.232	3.091		0.124
4	$CPC(0,1,1,2,1.5,1,0.8)$	0.757	1.045		0.059
5	$SU(0.1,1)$	0.877	34.644		0.104
6	$NGUM(0,1,0,1,0.5)$	0.889	2.496		0.048
7	$CGU(-0.8,1,2,7,1,0.8)$	0.953	0.574	A2	0.112
8	$NGUM(0,1,3,1,0.7)$	0.787	0.309		0.107
9	$CND(0,1,5,2,0.7)$	1.043	0.183		0.173
10	$NDPC(0,1,5,1.5,1,0.75)$	1.037	0.108		0.174
11	$SNM(-1,1,1,2,1.5,1,0.75)$	1.007	0.396		0.136
12	$CPC(0,1,1.5,1.5,1,0.8)$	1.247	0.825		0.175
13	$NDPC(-0.5,4,2,0,0.5,1,0.5)$	-0.241	2.795	B1	0.165
14	$SNM(0,0.5,1,-0.5,1,0.5,0.6)$	-0.59	1.582		0.068

Table 4. Continued

No.	Distribution	Skewness	Ex. kurtosis	Group	M
15	CND(0,1,-2,3,0.7)	-1.232	3.091		0.124
16	CPC(0,0.8,1,-2,2,1,0.7)	-1.25	2.102		0.121
17	CND(0.7,0.01,-0.05,1,0.07)	-0.163	0.227		0.044
18	NDPC(0.5,0.05,-0.1,1,1,0.1)	-0.162	0.137	B2	0.051
19	<i>CGU</i> (-3.8,1,-0.3,1,0.2)	-0.25	0.574		0.087
20	CPC(0,1,1,-5,1.5,1,0.8)	-1.247	0.825		0.175
21	SU(0,1)		33.188		0.099
22	CND(0,1,0,3,0.7)		3.488		0.074
23	CND(0,1,0,0.35,0.7)	0	0.894	C1	0.042
24	LAP(0,1)		3		0.062
25	t(5)		6		0.038
26	t(8)		1.5		0.021
27	<i>CL</i> (-0.75,1,0.75,1,0.5)		1.731		0.032
28	<i>CL</i> (-0.5,1,0.5,1,0.5)	0	2.346	C2	0.045
29	<i>CL</i> (-0.25,1,0.25,1,0.5)		2.819		0.057
30	CND(-2,2.5,2,2.5,0.5)		-0.305		0.011
31	CND(-0.8,1.5,0.8,1.5,0.5)		-0.098		0.003
32	<i>U</i> (-2,2)		-1.2		0.057
33	CND(-3,3,3,3,0.5)	0	-0.5	D1	0.022
34	TSN(-1.25,1.25)		-0.978		0.037
35	<i>U</i> (-1.5,1.5)		-1.2		0.057
36	TSN(-1.75,1.75)		-0.761		0.023
37	NDPC(-1.25,0.9,1.25,0.9,1,0.5)		-0.867		0.050
38	CND(-1,0.5,1,0.5,0.5)	0	-1.28	D2	0.099
39	<i>SNM</i> (-1.1,1,0.1,1,1,1,0.1,0.5)		-0.603		0.029
40	CPC(-1.05,1,1,1.05,1,1,0.5)		-0.55		0.025
41	CND(0.2,1,2.8,1,0.7)	0.392	-0.426		0.061
42	NDPC(0.1,1,2.8,1,1,0.7)	0.411	-0.453	E1	0.066
43	CPC(0.2,1,1,2.8,1,1,0.75)	0.483	-0.208		0.063
44	<i>SNM</i> (-1.5,1,1,0,1,0.5,0.6)	0.267	-0.17		0.027
45	<i>NGUM</i> (0,1,4,1,0.7)	0.804	-0.182		0.152
46	CND(0,1,4,1,0.7)	0.591	-0.735		0.129
47	<i>CGU</i> (-0.3,0.65,2.2,0.65,0.7)	0.694	-0.242	E2	0.116
48	CND(-0.2,1,3,1,0.7)	0.492	-0.577		0.091
49	CPC(-0.2,1,1,2.65,1,1,0.7)	0.437	-0.492		0.073
50	<i>SNM</i> (-1,0.8,1,1,0.8,2,0.7)	0.442	-0.71		0.094
51	CND(-0.2,0.5,-1.5,0.5,0.7)	-0.392	-0.426		0.061
52	CPC(-0.2,0.5,1,-1.5,0.5,1,0.75)	-0.483	-0.208	F1	0.063
53	NDPC(-0.2,0.6,-1.5,0.6,1,0.77)	-0.516	-0.105		0.043
54	<i>SNM</i> (1,1,-1.2,-0.7,1,-1.2,0.6)	-0.221	-0.485		0.039
55	CND(0,1,-4,1,0.7)	-0.591	-0.735		0.129
56	<i>CGU</i> (0,0.9,-3.8,1.25,0.7)	-0.439	-0.065		0.119
57	<i>NWM</i> (-1,1,1,3,0.7)	-0.274	-0.819	F2	0.091
58	CPC(-3,1,1,0,1,1,0.25)	-0.575	-0.263		0.083
59	<i>NGUM</i> (0,1,-3.4,1,0.75)	-0.585	-0.26		0.086
60	<i>SNM</i> (1,1,-1.2,-1,1,-1.2,0.6)	-0.242	-0.657		0.055

Table 8. PoT for the $LF(\bar{\alpha}, \bar{\beta})$ and $TCLF(\bar{\alpha}, \bar{\beta})$ tests, $n = 20, 50, 100$, alternatives 10–18

No	n	LF($\bar{\alpha}, \bar{\beta}$)										TCLF($\bar{\alpha}, \bar{\beta}$)									
		(0,1)	(1,0)	(0,0)	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{3}{10}, \frac{3}{10}\right)$	(1,1)	$\left(\frac{3}{8}, \frac{3}{8}\right)$	(0,1)	(1,0)	(0,0)	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{3}{10}, \frac{3}{10}\right)$	(1,1)	$\left(\frac{3}{8}, \frac{3}{8}\right)$						
10 _{A2}	20	0.835	0.466	0.743	0.748	0.754	0.742	0.752	0.771	0.622	0.739	0.754	0.747	0.741	0.748						
	50	0.998	0.986	0.995	0.996	0.996	0.996	0.996	0.996	0.987	0.995	0.995	0.995	0.995	0.995						
	100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000						
11 _{A2}	20	0.623	0.243	0.505	0.509	0.516	0.505	0.515	0.539	0.393	0.498	0.516	0.508	0.505	0.509						
	50	0.955	0.851	0.927	0.929	0.928	0.929	0.929	0.938	0.864	0.917	0.922	0.919	0.923	0.920						
	100	1.000	0.998	0.999	0.999	0.999	0.999	0.999	0.999	0.997	0.999	0.999	0.999	0.999	0.999						
12 _{A2}	20	0.825	0.476	0.727	0.745	0.745	0.754	0.746	0.762	0.656	0.738	0.753	0.746	0.745	0.747						
	50	0.996	0.982	0.992	0.993	0.993	0.994	0.993	0.995	0.987	0.993	0.994	0.993	0.994	0.993						
	100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000						
13 _{B1}	20	0.690	0.731	0.715	0.756	0.747	0.788	0.751	0.709	0.717	0.758	0.762	0.760	0.763	0.761						
	50	0.990	0.991	0.989	0.992	0.991	0.994	0.991	0.990	0.990	0.992	0.993	0.993	0.994	0.993						
	100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000						
14 _{B1}	20	0.090	0.231	0.155	0.178	0.173	0.200	0.176	0.162	0.192	0.183	0.184	0.182	0.192	0.183						
	50	0.262	0.425	0.337	0.368	0.355	0.396	0.361	0.359	0.411	0.388	0.403	0.394	0.413	0.398						
	100	0.555	0.681	0.606	0.634	0.622	0.650	0.627	0.637	0.684	0.658	0.675	0.668	0.688	0.671						
15 _{B1}	20	0.249	0.512	0.398	0.432	0.425	0.462	0.429	0.401	0.448	0.435	0.440	0.437	0.452	0.439						
	50	0.728	0.852	0.794	0.817	0.808	0.834	0.812	0.805	0.840	0.825	0.836	0.831	0.844	0.833						
	100	0.970	0.986	0.977	0.981	0.980	0.983	0.980	0.981	0.986	0.983	0.985	0.984	0.986	0.984						
16 _{B1}	20	0.208	0.521	0.397	0.421	0.419	0.440	0.420	0.366	0.437	0.419	0.426	0.423	0.434	0.424						
	50	0.712	0.864	0.805	0.820	0.814	0.832	0.817	0.774	0.840	0.815	0.826	0.821	0.832	0.824						
	100	0.972	0.989	0.982	0.984	0.983	0.985	0.984	0.977	0.987	0.983	0.984	0.984	0.986	0.984						
17 _{B2}	20	0.044	0.096	0.077	0.071	0.077	0.067	0.075	0.060	0.092	0.076	0.074	0.074	0.073	0.074						
	50	0.075	0.144	0.124	0.116	0.119	0.109	0.118	0.086	0.148	0.121	0.116	0.115	0.111	0.115						
	100	0.149	0.228	0.202	0.195	0.197	0.181	0.196	0.152	0.240	0.209	0.198	0.201	0.188	0.202						
18 _{B2}	20	0.054	0.133	0.103	0.099	0.104	0.095	0.103	0.075	0.118	0.101	0.103	0.101	0.102	0.101						
	50	0.128	0.224	0.190	0.187	0.188	0.182	0.188	0.138	0.218	0.184	0.184	0.182	0.183	0.183						
	100	0.280	0.373	0.337	0.337	0.336	0.326	0.337	0.274	0.369	0.333	0.331	0.332	0.329	0.333						

Table 9. PoT for the $LF(\bar{\alpha}, \bar{\beta})$ and $TCLF(\bar{\alpha}, \bar{\beta})$ tests, $n = 20, 50, 100$, alternatives 19–27

No	n	LF($\bar{\alpha}, \bar{\beta}$)										TCLF($\bar{\alpha}, \bar{\beta}$)									
		(0,1)	(1,0)	(0,0)	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{10}, \frac{3}{10})$	(1,1)	$(\frac{3}{8}, \frac{3}{8})$	(0,1)	(1,0)	(0,0)	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{10}, \frac{3}{10})$	(1,1)	$(\frac{3}{8}, \frac{3}{8})$						
19 _{B2}	20	0.104	0.270	0.178	0.208	0.201	0.236	0.205	0.190	0.219	0.214	0.213	0.212	0.219	0.213						
	50	0.370	0.581	0.468	0.510	0.493	0.546	0.501	0.505	0.558	0.543	0.554	0.547	0.558	0.550						
	100	0.786	0.888	0.830	0.854	0.844	0.866	0.848	0.855	0.887	0.873	0.882	0.878	0.889	0.880						
20 _{B2}	20	0.480	0.824	0.727	0.744	0.745	0.753	0.745	0.661	0.756	0.740	0.747	0.744	0.749	0.746						
	50	0.982	0.996	0.992	0.993	0.993	0.994	0.993	0.987	0.994	0.992	0.993	0.993	0.994	0.993						
	100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000						
21 _{C1}	20	0.314	0.317	0.304	0.343	0.333	0.381	0.338	0.350	0.349	0.355	0.356	0.353	0.367	0.354						
	50	0.629	0.628	0.608	0.645	0.630	0.678	0.637	0.679	0.679	0.683	0.694	0.688	0.702	0.690						
	100	0.887	0.888	0.875	0.893	0.886	0.904	0.889	0.916	0.916	0.916	0.922	0.920	0.927	0.921						
22 _{C1}	20	0.248	0.246	0.229	0.273	0.261	0.316	0.266	0.288	0.286	0.288	0.288	0.284	0.304	0.286						
	50	0.518	0.516	0.487	0.536	0.516	0.582	0.525	0.597	0.595	0.596	0.611	0.601	0.624	0.604						
	100	0.814	0.813	0.789	0.822	0.809	0.842	0.814	0.873	0.873	0.872	0.882	0.878	0.890	0.880						
23 _{C1}	20	0.110	0.107	0.094	0.114	0.110	0.133	0.112	0.115	0.113	0.118	0.119	0.117	0.123	0.118						
	50	0.194	0.192	0.174	0.204	0.192	0.231	0.197	0.224	0.225	0.231	0.241	0.235	0.246	0.237						
	100	0.350	0.351	0.323	0.358	0.343	0.382	0.349	0.404	0.402	0.405	0.423	0.416	0.435	0.420						
24 _{C1}	20	0.200	0.198	0.185	0.219	0.210	0.251	0.214	0.219	0.217	0.227	0.228	0.225	0.236	0.227						
	50	0.412	0.411	0.389	0.432	0.414	0.469	0.421	0.459	0.459	0.469	0.483	0.474	0.491	0.477						
	100	0.697	0.697	0.673	0.707	0.693	0.729	0.699	0.742	0.743	0.744	0.760	0.754	0.771	0.756						
25 _{C1}	20	0.121	0.120	0.111	0.129	0.125	0.149	0.127	0.139	0.138	0.135	0.136	0.134	0.144	0.135						
	50	0.200	0.201	0.183	0.210	0.198	0.235	0.203	0.246	0.246	0.242	0.252	0.245	0.262	0.248						
	100	0.325	0.325	0.300	0.331	0.318	0.352	0.323	0.398	0.399	0.393	0.408	0.402	0.421	0.405						
26 _{C1}	20	0.079	0.080	0.071	0.082	0.080	0.093	0.081	0.088	0.087	0.085	0.087	0.085	0.091	0.085						
	50	0.106	0.106	0.097	0.111	0.105	0.125	0.108	0.131	0.130	0.127	0.134	0.129	0.140	0.130						
	100	0.144	0.143	0.128	0.145	0.138	0.157	0.141	0.187	0.185	0.182	0.191	0.187	0.201	0.189						
27 _{C2}	20	0.094	0.095	0.087	0.099	0.097	0.111	0.098	0.109	0.110	0.105	0.106	0.104	0.113	0.104						
	50	0.134	0.133	0.121	0.140	0.132	0.159	0.135	0.180	0.178	0.169	0.177	0.171	0.188	0.172						
	100	0.203	0.203	0.182	0.208	0.197	0.226	0.202	0.288	0.290	0.281	0.291	0.287	0.303	0.290						

Table 10. PoT for the $LF(\bar{\alpha}, \bar{\beta})$ and $TCLF(\bar{\alpha}, \bar{\beta})$ tests, $n = 20, 50, 100$, alternatives 28–36

No	n	LF($\bar{\alpha}, \bar{\beta}$)						TCLF($\bar{\alpha}, \bar{\beta}$)							
		(0,1)	(1,0)	(0,0)	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{3}{10}, \frac{3}{10}\right)$	(1,1)	$\left(\frac{3}{8}, \frac{3}{8}\right)$	(0,1)	(1,0)	(0,0)	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{3}{10}, \frac{3}{10}\right)$	(1,1)	$\left(\frac{3}{8}, \frac{3}{8}\right)$
28 _{c2}	20	0.127	0.128	0.114	0.136	0.131	0.160	0.133	0.147	0.146	0.145	0.145	0.143	0.154	0.144
	50	0.227	0.226	0.204	0.238	0.224	0.269	0.230	0.282	0.281	0.278	0.290	0.281	0.301	0.284
	100	0.394	0.394	0.364	0.403	0.386	0.431	0.393	0.484	0.484	0.481	0.497	0.492	0.510	0.495
29 _{c2}	20	0.174	0.174	0.160	0.191	0.183	0.221	0.187	0.195	0.193	0.199	0.200	0.198	0.209	0.199
	50	0.355	0.354	0.329	0.370	0.353	0.408	0.360	0.406	0.406	0.411	0.426	0.417	0.435	0.420
	100	0.612	0.612	0.585	0.621	0.606	0.646	0.612	0.673	0.672	0.673	0.690	0.684	0.701	0.686
30 _{b1}	20	0.052	0.051	0.058	0.050	0.055	0.042	0.053	0.056	0.054	0.052	0.051	0.051	0.046	0.050
	50	0.066	0.066	0.077	0.067	0.071	0.056	0.070	0.064	0.066	0.067	0.058	0.060	0.050	0.060
	100	0.100	0.099	0.112	0.101	0.105	0.085	0.104	0.095	0.095	0.104	0.088	0.094	0.075	0.092
31 _{b1}	20	0.048	0.048	0.047	0.046	0.049	0.047	0.048	0.050	0.049	0.048	0.048	0.048	0.047	0.047
	50	0.047	0.047	0.048	0.048	0.047	0.047	0.048	0.047	0.048	0.048	0.047	0.046	0.047	0.047
	100	0.049	0.049	0.049	0.049	0.049	0.046	0.049	0.047	0.047	0.048	0.046	0.047	0.046	0.047
32 _{b1}	20	0.097	0.099	0.130	0.095	0.113	0.068	0.107	0.133	0.133	0.116	0.098	0.103	0.081	0.099
	50	0.254	0.252	0.315	0.261	0.282	0.207	0.275	0.296	0.297	0.325	0.251	0.273	0.193	0.265
	100	0.580	0.579	0.639	0.588	0.607	0.519	0.601	0.631	0.631	0.695	0.615	0.647	0.531	0.635
33 _{A1}	20	0.049	0.049	0.053	0.047	0.052	0.042	0.050	0.052	0.051	0.049	0.048	0.048	0.045	0.048
	50	0.056	0.057	0.065	0.057	0.060	0.050	0.059	0.055	0.056	0.057	0.050	0.052	0.045	0.051
	100	0.074	0.074	0.082	0.074	0.077	0.063	0.076	0.068	0.069	0.074	0.063	0.067	0.055	0.066
34 _{b1}	20	0.056	0.055	0.067	0.052	0.061	0.040	0.058	0.070	0.068	0.059	0.054	0.055	0.046	0.053
	50	0.100	0.098	0.123	0.099	0.108	0.078	0.105	0.116	0.116	0.122	0.090	0.099	0.070	0.096
	100	0.208	0.208	0.242	0.210	0.222	0.171	0.218	0.241	0.244	0.276	0.216	0.238	0.168	0.229
35 _{b1}	20	0.098	0.099	0.131	0.098	0.115	0.070	0.109	0.135	0.134	0.117	0.101	0.105	0.084	0.102
	50	0.253	0.252	0.316	0.260	0.281	0.207	0.274	0.295	0.296	0.325	0.250	0.273	0.192	0.265
	100	0.579	0.578	0.638	0.584	0.606	0.515	0.599	0.629	0.628	0.693	0.613	0.645	0.528	0.631
36 _{b1}	20	0.047	0.044	0.052	0.043	0.049	0.037	0.047	0.052	0.048	0.047	0.045	0.045	0.039	0.044
	50	0.055	0.055	0.065	0.055	0.059	0.046	0.058	0.059	0.060	0.060	0.049	0.052	0.041	0.051
	100	0.085	0.085	0.098	0.085	0.090	0.069	0.088	0.093	0.094	0.104	0.080	0.088	0.063	0.085

Table 11. PoT for the $LF(\bar{\alpha}, \bar{\beta})$ and $TCLF(\bar{\alpha}, \bar{\beta})$ tests, $n = 20, 50, 100$, alternatives 37–45

No	n	LF($\bar{\alpha}, \bar{\beta}$)							TCLF($\bar{\alpha}, \bar{\beta}$)						
		(0,1)	(1,0)	(0,0)	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{10}, \frac{3}{10})$	(1,1)	$(\frac{3}{8}, \frac{3}{8})$	(0,1)	(1,0)	(0,0)	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{10}, \frac{3}{10})$	(1,1)	$(\frac{3}{8}, \frac{3}{8})$
37 _{D2}	20	0.095	0.094	0.121	0.096	0.110	0.073	0.105	0.108	0.107	0.104	0.098	0.099	0.085	0.097
	50	0.216	0.214	0.260	0.224	0.238	0.187	0.234	0.214	0.215	0.233	0.201	0.210	0.170	0.207
	100	0.460	0.460	0.503	0.469	0.482	0.419	0.477	0.445	0.446	0.486	0.441	0.458	0.395	0.453
38 _{D2}	20	0.321	0.316	0.413	0.347	0.383	0.277	0.371	0.357	0.350	0.374	0.353	0.359	0.311	0.354
	50	0.805	0.806	0.860	0.828	0.842	0.782	0.838	0.789	0.792	0.834	0.802	0.813	0.758	0.810
	100	0.994	0.994	0.996	0.995	0.995	0.992	0.995	0.992	0.992	0.995	0.993	0.994	0.990	0.993
39 _{D2}	20	0.057	0.057	0.068	0.056	0.063	0.045	0.060	0.063	0.061	0.059	0.057	0.057	0.051	0.057
	50	0.086	0.086	0.103	0.088	0.094	0.074	0.092	0.085	0.087	0.090	0.077	0.080	0.065	0.079
	100	0.150	0.150	0.170	0.152	0.159	0.129	0.157	0.144	0.145	0.160	0.135	0.144	0.115	0.142
40 _{D2}	20	0.055	0.054	0.063	0.053	0.059	0.045	0.057	0.060	0.058	0.056	0.054	0.054	0.049	0.054
	50	0.074	0.073	0.088	0.075	0.079	0.062	0.078	0.073	0.074	0.076	0.064	0.067	0.055	0.067
	100	0.120	0.120	0.137	0.122	0.128	0.102	0.126	0.115	0.115	0.127	0.106	0.114	0.091	0.112
41 _{E1}	20	0.183	0.041	0.128	0.121	0.128	0.113	0.126	0.145	0.084	0.120	0.124	0.122	0.117	0.122
	50	0.356	0.167	0.290	0.282	0.285	0.269	0.285	0.309	0.185	0.262	0.260	0.258	0.254	0.259
	100	0.613	0.445	0.554	0.549	0.550	0.530	0.550	0.561	0.421	0.511	0.501	0.505	0.494	0.505
42 _{E1}	20	0.204	0.046	0.144	0.137	0.145	0.128	0.142	0.163	0.094	0.135	0.140	0.138	0.131	0.137
	50	0.403	0.200	0.336	0.327	0.331	0.313	0.330	0.355	0.217	0.305	0.302	0.300	0.295	0.301
	100	0.683	0.519	0.627	0.622	0.624	0.604	0.624	0.633	0.493	0.586	0.576	0.580	0.570	0.580
43 _{E1}	20	0.205	0.040	0.134	0.134	0.138	0.131	0.137	0.156	0.091	0.131	0.138	0.134	0.131	0.134
	50	0.392	0.186	0.312	0.312	0.311	0.308	0.312	0.339	0.213	0.291	0.296	0.291	0.294	0.293
	100	0.658	0.485	0.589	0.592	0.590	0.581	0.592	0.604	0.468	0.552	0.551	0.551	0.553	0.553
44 _{E1}	20	0.088	0.030	0.062	0.059	0.063	0.057	0.062	0.074	0.048	0.060	0.061	0.060	0.058	0.060
	50	0.125	0.048	0.094	0.091	0.092	0.087	0.092	0.107	0.058	0.086	0.085	0.084	0.083	0.085
	100	0.184	0.096	0.146	0.144	0.145	0.136	0.145	0.158	0.092	0.129	0.125	0.127	0.124	0.127
45 _{E2}	20	0.711	0.327	0.621	0.605	0.621	0.578	0.616	0.643	0.456	0.600	0.611	0.606	0.589	0.605
	50	0.987	0.942	0.980	0.978	0.979	0.974	0.979	0.982	0.937	0.973	0.972	0.972	0.970	0.972
	100	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 12. PoT for the $LF(\bar{\alpha}, \bar{\beta})$ and $TCLF(\bar{\alpha}, \bar{\beta})$ tests, $n = 20, 50, 100$, alternatives 46–54

No	n	LF($\bar{\alpha}, \bar{\beta}$)						TCLF($\bar{\alpha}, \bar{\beta}$)							
		(0,1)	(1,0)	(0,0)	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{10}, \frac{3}{10})$	(1,1)	$(\frac{3}{8}, \frac{3}{8})$	(0,1)	(1,0)	(0,0)	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{10}, \frac{3}{10})$	(1,1)	$(\frac{3}{8}, \frac{3}{8})$
46 _{E2}	20	0.569	0.220	0.477	0.457	0.475	0.429	0.469	0.495	0.352	0.456	0.463	0.459	0.439	0.458
	50	1.000	0.830	0.918	0.913	0.915	0.903	0.915	0.919	0.837	0.902	0.898	0.898	0.891	0.899
	100	0.999	0.997	0.999	0.999	0.999	0.999	0.999	0.999	0.996	0.998	0.998	0.998	0.998	0.998
47 _{E2}	20	0.473	0.159	0.398	0.364	0.386	0.323	0.379	0.439	0.250	0.368	0.370	0.368	0.355	0.366
	50	0.872	0.688	0.839	0.820	0.827	0.794	0.825	0.856	0.663	0.808	0.794	0.797	0.778	0.797
	100	0.995	0.982	0.993	0.992	0.992	0.990	0.992	0.994	0.974	0.990	0.989	0.990	0.987	0.989
48 _{E2}	20	0.324	0.085	0.244	0.232	0.243	0.217	0.240	0.265	0.164	0.231	0.237	0.233	0.222	0.232
	50	0.659	0.424	0.593	0.582	0.586	0.564	0.586	0.608	0.443	0.556	0.551	0.550	0.541	0.551
	100	0.929	0.852	0.908	0.904	0.905	0.894	0.905	0.907	0.832	0.887	0.881	0.884	0.876	0.884
49 _{E2}	20	0.236	0.053	0.168	0.160	0.168	0.148	0.166	0.189	0.110	0.158	0.164	0.160	0.152	0.160
	50	0.480	0.258	0.409	0.400	0.404	0.385	0.403	0.429	0.276	0.376	0.373	0.370	0.365	0.372
	100	0.777	0.629	0.730	0.725	0.727	0.708	0.727	0.734	0.603	0.692	0.682	0.686	0.675	0.686
50 _{E2}	20	0.337	0.103	0.268	0.248	0.263	0.223	0.259	0.284	0.184	0.250	0.253	0.250	0.235	0.249
	50	0.695	0.483	0.647	0.628	0.636	0.602	0.634	0.650	0.496	0.610	0.596	0.598	0.578	0.598
	100	0.950	0.897	0.938	0.933	0.935	0.923	0.934	0.933	0.879	0.923	0.915	0.918	0.908	0.918
51 _{F1}	20	0.040	0.182	0.127	0.120	0.127	0.112	0.125	0.086	0.142	0.120	0.122	0.121	0.118	0.120
	50	0.167	0.356	0.289	0.280	0.284	0.268	0.284	0.181	0.311	0.260	0.257	0.256	0.251	0.257
	100	0.444	0.613	0.553	0.547	0.549	0.529	0.549	0.417	0.561	0.510	0.500	0.505	0.496	0.505
52 _{F1}	20	0.041	0.202	0.135	0.134	0.139	0.131	0.138	0.092	0.152	0.133	0.135	0.134	0.134	0.134
	50	0.189	0.393	0.316	0.316	0.315	0.311	0.316	0.215	0.343	0.294	0.296	0.293	0.296	0.296
	100	0.486	0.658	0.592	0.593	0.592	0.582	0.593	0.468	0.606	0.553	0.553	0.553	0.555	0.553
53 _{F1}	20	0.030	0.133	0.085	0.086	0.089	0.086	0.088	0.064	0.099	0.086	0.087	0.086	0.087	0.086
	50	0.089	0.222	0.166	0.168	0.167	0.167	0.168	0.109	0.189	0.156	0.159	0.157	0.160	0.159
	100	0.224	0.372	0.307	0.311	0.309	0.304	0.311	0.220	0.328	0.282	0.284	0.284	0.288	0.284
54 _{F1}	20	0.037	0.094	0.075	0.064	0.071	0.055	0.069	0.055	0.083	0.067	0.066	0.066	0.061	0.065
	50	0.072	0.154	0.133	0.120	0.124	0.105	0.123	0.075	0.143	0.115	0.105	0.107	0.095	0.107
	100	0.162	0.266	0.237	0.221	0.227	0.197	0.226	0.144	0.246	0.214	0.193	0.200	0.176	0.199

Table 13. PoT for the $LF(\bar{\alpha}, \bar{\beta})$ and $TCLF(\bar{\alpha}, \bar{\beta})$ tests, $n = 20, 50, 100$, alternatives 55–60

No	n	$LF(\bar{\alpha}, \bar{\beta})$						$TCLF(\bar{\alpha}, \bar{\beta})$							
		(0,1)	(1,0)	(0,0)	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{10}, \frac{3}{10})$	(1,1)	$(\frac{3}{8}, \frac{3}{8})$	(0,1)	(1,0)	(0,0)	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{10}, \frac{3}{10})$	(1,1)	$(\frac{3}{8}, \frac{3}{8})$
55 _{F2}	20	0.221	0.568	0.476	0.458	0.475	0.430	0.470	0.356	0.486	0.459	0.461	0.459	0.445	0.458
	50	0.832	0.940	0.920	0.914	0.916	0.903	0.916	0.835	0.921	0.903	0.898	0.899	0.891	0.900
	100	0.997	0.999	0.999	0.999	0.999	0.998	0.999	0.996	0.999	0.998	0.998	0.998	0.998	0.998
56 _{F2}	20	0.165	0.467	0.347	0.365	0.366	0.378	0.367	0.299	0.371	0.364	0.368	0.366	0.365	0.366
	50	0.684	0.857	0.794	0.807	0.802	0.814	0.804	0.742	0.818	0.797	0.804	0.799	0.805	0.802
	100	0.978	0.992	0.987	0.988	0.988	0.988	0.988	0.979	0.989	0.986	0.987	0.986	0.987	0.986
57 _{F2}	20	0.085	0.212	0.203	0.159	0.183	0.120	0.174	0.103	0.263	0.182	0.164	0.170	0.141	0.165
	50	0.328	0.531	0.515	0.460	0.481	0.399	0.474	0.279	0.598	0.515	0.450	0.470	0.384	0.462
	100	0.768	0.877	0.866	0.841	0.851	0.804	0.847	0.739	0.911	0.884	0.850	0.864	0.809	0.860
58 _{F2}	20	0.067	0.297	0.207	0.207	0.213	0.201	0.212	0.144	0.226	0.204	0.208	0.206	0.205	0.206
	50	0.355	0.599	0.516	0.517	0.516	0.511	0.518	0.384	0.544	0.489	0.492	0.489	0.491	0.492
	100	0.767	0.881	0.842	0.843	0.842	0.836	0.843	0.750	0.850	0.816	0.815	0.816	0.816	0.816
59 _{F2}	20	0.075	0.318	0.220	0.223	0.228	0.223	0.227	0.166	0.242	0.221	0.225	0.223	0.223	0.223
	50	0.386	0.633	0.543	0.548	0.546	0.548	0.548	0.433	0.576	0.526	0.531	0.526	0.532	0.530
	100	0.807	0.906	0.870	0.873	0.872	0.870	0.873	0.805	0.881	0.853	0.855	0.855	0.858	0.855
60 _{F2}	20	0.049	0.131	0.109	0.093	0.104	0.077	0.100	0.075	0.119	0.098	0.095	0.096	0.087	0.094
	50	0.141	0.265	0.238	0.215	0.224	0.189	0.221	0.138	0.250	0.210	0.192	0.196	0.172	0.195
	100	0.349	0.488	0.456	0.432	0.441	0.395	0.438	0.317	0.463	0.424	0.392	0.403	0.364	0.402

Table 14. The analyzed GoFTs with the highest power for groups (G) A1-F2, $n = 20, 50, 100$ and alternatives (A) 1 – 60 ($O(\bar{\alpha}, \bar{\beta})$ — old i.e. : $LF(\bar{\alpha}, \bar{\beta})$ test is at least 0.002 more powerful, $N(\bar{\alpha}, \bar{\beta})$ — new i.e. $TCLF(\bar{\alpha}, \bar{\beta})$ test is at least 0.002 more powerful, $ON(\bar{\alpha}, \bar{\beta})$ — difference between PoTs is less than 0.002)

G	A	n=20	n=50	n=100	G	A	n=20	n=50	n=100
A1	1	O(0,1)	O(0,1)	O(0,1)	D1	31	ON(0,1)	ON(0,0)	ON(0,0)
	2	O(0,1)	O(0,1)	O(0,1)		32	N(0,1)	N(0,0)	N(0,0)
	3	O(0,1)	O(0,1)	ON(1,1)		33	ON(0,1)	O(0,0)	O(0,0)
	4	O(0,1)	O(0,1)	O(0,1)		34	N(0,1)	ON(0,0)	N(0,0)
	5	O(1,1)	N(1,1)	N(1,1)		35	N(0,1)	N(0,0)	N(0,0)
	6	O(0,1)	O(0,1)	O(0,1)		36	ON(0,1)	O(0,0)	N(0,0)
A2	7	O(0,1)	O(0,1)	O(0,1)	D2	37	O(0,0)	O(0,0)	O(0,0)
	8	O(0,1)	O(0,1)	O(0,1)		38	O(0,0)	O(0,0)	ON(0,0)
	9	O(0,1)	ON(0,1)	ON(0,1)		39	O(0,0)	O(0,0)	O(0,0)
	10	O(0,1)	ON(0,1)	ON(0,1)		40	O(0,0)	O(0,0)	O(0,0)
	11	O(0,1)	O(1,1)	ON(1,1)		41	O(0,1)	O(0,1)	O(0,1)
	12	O(0,1)	ON(0,1)	ON(0,1)		42	O(0,1)	O(0,1)	O(0,1)
B1	13	O(1,1)	ON(1,1)	ON(1,1)	E1	43	O(0,1)	O(0,1)	O(0,1)
	14	O(1,0)	O(1,0)	N(1,1)		44	O(0,1)	O(0,1)	O(0,1)
	15	O(1,0)	O(1,0)	ON(1,1)		45	O(0,1)	O(0,1)	ON(0,1)
	16	O(1,0)	O(1,0)	O(1,0)		46	O(0,1)	O(0,1)	ON(0,1)
B2	17	O(1,0)	N(1,0)	N(1,0)	E2	47	O(0,1)	O(0,1)	ON(0,1)
	18	O(1,0)	O(1,0)	O(1,0)		48	O(0,1)	O(0,1)	O(0,1)
	19	O(1,0)	O(1,0)	ON(1,0)		49	O(0,1)	O(0,1)	O(0,1)
	20	O(1,0)	ON(1,0)	ON(1,0)		50	O(0,1)	O(0,1)	O(0,1)
C1	21	O(1,1)	N(1,1)	N(1,1)	F1	51	O(1,0)	O(1,0)	O(1,0)
	22	O(1,1)	N(1,1)	N(1,1)		52	O(1,0)	O(1,0)	O(1,0)
	23	O(1,1)	N(1,1)	N(1,1)		53	O(1,0)	O(1,0)	O(1,0)
	24	O(1,1)	N(1,1)	N(1,1)		54	O(1,0)	O(1,0)	O(1,0)
	25	O(1,1)	N(1,1)	N(1,1)		55	O(1,0)	O(1,0)	ON(1,0)
	26	O(1,1)	N(1,1)	N(1,1)		56	O(1,0)	O(1,0)	O(1,0)
C2	27	ON(1,1)	N(1,1)	N(1,1)	F2	57	N(1,0)	N(1,0)	N(1,0)
	28	O(1,1)	N(1,1)	N(1,1)		58	O(1,0)	O(1,0)	O(1,0)
	29	O(1,1)	N(1,1)	N(1,1)		59	O(1,0)	O(1,0)	O(1,0)
D1	30	O(0,0)	O(0,0)	O(0,0)		60	O(1,0)	O(1,0)	O(1,0)

Table 15. $LF(\bar{\alpha}, \bar{\beta})$ and $TCLF(\bar{\alpha}, \bar{\beta})$ tests, p-values, examples I–V

Ex	n	Test	(0,1)	(1,0)	(0,0)	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{3}{10}, \frac{3}{10}\right)$	(1,1)	$\left(\frac{3}{8}, \frac{3}{8}\right)$
I	99	$LF(\bar{\alpha}, \bar{\beta})$	0.428	0.308	0.471	0.406	0.431	0.353	0.421
		$TCLF(\bar{\alpha}, \bar{\beta})$	0.403	0.283	0.395	0.327	0.349	0.291	0.342
II	74	$LF(\bar{\alpha}, \bar{\beta})$	0.708	0.836	0.889	0.829	0.864	0.740	0.851
		$TCLF(\bar{\alpha}, \bar{\beta})$	0.656	0.853	0.903	0.724	0.768	0.658	0.748
III	15	$LF(\bar{\alpha}, \bar{\beta})$	0.900	0.964	0.957	0.983	0.975	0.906	0.978
		$TCLF(\bar{\alpha}, \bar{\beta})$	0.932	0.887	0.966	0.987	0.982	0.685	0.984
IV	50	$LF(\bar{\alpha}, \bar{\beta})$	0.022	0.087	0.027	0.036	0.032	0.050	0.034
		$TCLF(\bar{\alpha}, \bar{\beta})$	0.019	0.129	0.035	0.043	0.040	0.056	0.041
V	50	$LF(\bar{\alpha}, \bar{\beta})$	0.804	0.411	0.442	0.556	0.508	0.692	0.526
		$TCLF(\bar{\alpha}, \bar{\beta})$	0.808	0.297	0.363	0.490	0.429	0.658	0.452

Table 16. Other GoFTs, p-values, examples I–V

Ex	n	AD	CVM	SW	SF	Hn	MAD	χ^2
I	99	0.338	0.315	0.341	0.472	0.453	0.347	0.453
II	74	0.751	0.763	0.729	0.738	0.822	0.894	0.765
III	15	0.873	0.889	0.698	0.882	0.896	0.691	0.881
IV	50	0.055	0.064	0.041	0.100	0.101	0.046	0.058
V	50	0.525	0.584	0.438	0.591	0.587	0.701	0.552

Figure 1. Histogram and normal distribution. Real-life examples

APPENDIX

```
TCLF <- function(x, alfa, beta) {
  x <- sort(x)
  n <- length(x)
  FT <- pnorm(x, mean(x), sd(x))
  cFn <- (1:n - alfa) / (n - alfa - beta + 1)
  poz <- which.max(abs(cFn - FT))
  return(list(stat.TCLF = max(abs(cFn - FT)),
             position = poz)) }

L.boot <- function(data, B, r, alfa, beta, conf.level = 0.95) {
  n <- length(data)
  counter <- 0
  stat.TCLF <- numeric(B)
  while (counter <= B) {
    x <- rnorm(n)
    result <- TCLF(x, alfa, beta)
    if (result$position == r) {
      counter <- counter + 1
      stat.TCLF[counter] <- result$stat.TCLF
    }
  }
  return(list(Q = as.numeric(quantile(stat.TCLF, conf.level)),
             p.value = mean(stat.TCLF > TCLF(data, alfa, beta)$stat.TCLF)))
}

alfa <- 0
beta <- 1
B <- 1e4
#example I
data=c(96.1,97.1,97.1,97.2,99.2,99.4,99.4,99.9,100.4,101,101.1,101.5,
102.4,102.9,103,103.3,103.3,103.5,104.1,104.6,104.6,104.8,105.4,105.4,
105.6,105.7,105.8,105.9,105.9,105.9,105.9,106.2,106.2,106.3,106.3,106.7,
106.8,106.9,106.9,107.1,107.4,107.6,107.7,108,108.1,108.2,108.2,108.5,
108.6,108.7,108.8,109.1,109.2,109.2,109.3,109.3,109.4,109.4,109.6,
109.6,110,110.1,110.1,110.1,110.2,110.3,110.4,110.5,110.7,110.8,110.8,
110.8,111,111.1,111.2,111.4,111.8,112,112.1,112.7,112.8,112.9,113.3,
113.7,114.3,114.9,114.9,115.3,115.5,115.9,116.3,116.7,117,117.9,117.9,
119.2,119.3,119.5)
#example II
#data=c(1.312,1.314,1.479,1.552,1.7,1.803,1.861,1.865,1.944,1.958,1.966,
1.997,2.006,2.021,2.027,2.055,2.063,2.098,2.14,2.179,2.224,2.24,2.253,
2.27,2.272,2.274,2.301,2.301,2.359,2.382,2.382,2.426,2.434,2.435,2.478,
2.49,2.511,2.514,2.535,2.554,2.566,2.57,2.586,2.629,2.633,2.642,2.648,
```

```
2.684,2.697,2.726,2.77,2.773,2.8,2.809,2.818,2.821,2.848,2.88,2.809,2.818  
,2.821,2.848,2.88,2.954,3.012,3.067,3.084,3.09,3.096,3.128,3.233,3.433,  
3.585,3.585)  
#example III  
#data=t(women[2])  
#example IV  
#data=t(USArrests[2])  
#example V  
#data=t(USArrests[3])  
TCLF(data, alfa, beta)  
L.boot(data, B, TCLF(data, alfa, beta)$position, alfa, beta)
```